STABILITY ESTIMATES FOR THE INVERSE BOUNDARY VALUE PROBLEM BY PARTIAL CAUCHY DATA

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Abstract. In this paper we study the inverse conductivity problem with partial data in dimension $n \geq 3$. We derive stability estimates for this inverse problem if the conductivity has $C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ regularity for $0 < \sigma < 1$.

1. Introduction

In 1980, A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [6]. This pioneer's contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" (CGO) solutions of partial differential equations to solve inverse problems. The problem that Calderón considered was whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is known as *Electrical Impedance Tomography* (EIT). EIT arises not only in geophysical prospections (See [29]) but also in medical imaging (See [11], [12] and [15]). We now describe more precisely the mathematical problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The Dirichlet-to-Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega},$$

where $\partial_\nu u = \nu \cdot \nabla u$ and ν is the unit outer normal to $\partial\Omega$. The well-known inverse problem is to recover the conductivity γ from the boundary measurement Λ_γ.

The uniqueness issue for C^2 conductivities was settled by Sylvester and Uhlmann [23]. The regularity of conductivity was relaxed to $3/2$ derivatives in some sense in [4] and [20]. Uniqueness for conormal conductivities in $C^{1,\varepsilon}$ was shown in [8]. See [26] for the detailed development. Recently, Haberman and Tataru [9] extended the uniqueness result to C^1 conductivities or small in the $W^{1,\infty}$ norm. It is an open problem whether uniqueness holds in dimension $n \geq 3$ for Lipschitz or less regular conductivities.

For the stability result, in 1988, a log-type stability estimate was derived by Alessandrini [1]. Mandache [18] has shown that this estimate is optimal. Later, Heck [10] proved the stability for conductivities in $C^{1,\frac{1}{2}+\varepsilon} \cap H^{\frac{n}{2}+\varepsilon}$ with smooth boundary in 2009. For the case $\gamma \in C^{1,\varepsilon}, 0 < \varepsilon < 1$, Caro, García and Reyes used Haberman and Tataru's ideas to derive the stability result with Lipschitz boundary. For a review of stability issues in EIT see [3].

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All results mentioned above are concerned with the full data. In several applications in EIT one can only measure currents and voltages on part of the boundary. A general uniqueness result with partial data was first obtained by Bukhgeim and Uhlmann [5] when the Neumann data were taken on part of $\partial\Omega$ which is slightly larger than the half of the boundary. Their result was improved in [17] where the Cauchy data can be taken on any part of the boundary. In [5] and [17], the conductivities are in C^2. The regularity assumption on the conductivity was relaxed to $C^{1,\varepsilon}, \varepsilon > 0$ by Knudsen in [16]. In 2012, Zhang [28] gave the uniqueness result with $C^1 \cap H^{\frac{3}{2}}$ conductivities by using the idea in [9] and following the argument in [16]. The stability estimates for the uniqueness result of [5] were given by Heck and Wang in [13]. Heck and Wang proved the log-log type stability estimate with partial data. They improved their result to the log type stability in the paper [14] in 2007 by considering special domain. In this paper, we derive a log-log type stability estimate for less regular conductivities.

To state the main result, we first introduce some notations. Picking a $\eta \in S^{n-1}$ and letting $\varepsilon > 0$, we define

$$\partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega : \eta \cdot \nu(x) > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon} = \partial\Omega \backslash \overline{\partial\Omega_{+,\varepsilon}}.$$

The localized Dirichlet-to-Neumann map is given by

$$\tilde{\Lambda}_\gamma : f \mapsto \gamma \partial_\nu u|_{\partial\Omega_{-,\varepsilon}}.$$

So $\tilde{\Lambda}_\gamma$ is an operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $\tilde{H}^{-\frac{1}{2}}(\partial\Omega_{-,\varepsilon})$, the restriction of $H^{-\frac{1}{2}}(\partial\Omega)$ onto $\partial\Omega_{-,\varepsilon}$. The operator norm of $\tilde{\Lambda}_\gamma$ is denoted by $\|\tilde{\Lambda}_\gamma\|_*$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open, bounded domain with C^2 boundary. Assume that $\gamma_j \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ with $0 < \sigma < 1$ and $\gamma_j > \gamma_0 > 0$ for $j = 1, 2$. Suppose that*

$$\gamma_1 = \gamma_2 \quad \textit{and} \quad \partial_\nu \gamma_1 = \partial_\nu \gamma_2 \quad \textit{on } \overline{\partial\Omega_{+,\varepsilon}}.$$

Then there exist constants $\theta, \tilde{\theta}, \tilde{\sigma} \in (0, 1)$ such that

$$\|\gamma_1 - \gamma_2\|_{C^{0,\tilde{\sigma}}(\overline{\Omega})} \lesssim \Big(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^\theta + \frac{1}{K} \log|\log \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^\theta|^{\frac{-1}{\theta}} \Big)^{\frac{\tilde{\theta}(1-\tilde{\sigma})}{n}}. \tag{1.2}$$

Note that the symbol $\lesssim$ means that there exists a positive constant for which the estimate holds whenever the right hand side of the estimate is multiplied by that constant.

To deriving the estimate (1.2), we adapt Zhang's argument [28] to the case $\tilde{\Lambda}_{\gamma_1} \neq \tilde{\Lambda}_{\gamma_2}$. Then we will get an estimate of the Fourier transform of $q = (ik)\nabla v + \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})\nabla v$ on some subset of $\mathbb{R}^n$ where $v = \log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}$. Since q can be treated as a compactly supported function, its Fourier transform is real analytic. We use Vessella's stability estimate for analytic continuation [27] to our case here. This idea was first introduced in [13] to get the log-log type stability estimate with partial measurements.

2. Preliminary result

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary $\partial\Omega$ throughout the paper. Assume that $\gamma_j \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ with $0 < \sigma < 1$ and $\gamma_j > \gamma_0 > 0$ for $j = 1, 2$. Let $\overline{\Omega} \subset B$. We can extend γ_j to be the function in $\mathbb{R}^n$ such that $\gamma_j \in C^{1,\sigma}(\mathbb{R}^n)$ and $\gamma_j - 1 \in H^{\frac{3}{2}+\sigma}(\mathbb{R}^n)$ with $\mathrm{supp}(\gamma_j - 1) \subset \overline{B}$.

Let $\Psi_t = t^n \Psi(tx)$ where $\Psi \in C_0^\infty(\mathbb{R}^n)$ supported on the unit ball and $\int \Psi = 1$. Denote that $\phi = \log\gamma$ and $A = \nabla \log\gamma$. Define $\phi_t = \Psi_t * \phi$ and $A_t = \Psi_t * A$. Then the following results are from [16] and [21].

Lemma 2.1. *Let $\gamma \in C^{1,\sigma}(\mathbb{R}^n)$ for $0 \le \sigma \le 1$ and $\gamma - 1 \in H^{\frac{3}{2}+\sigma}(\mathbb{R}^n)$ with compact support. Then*

$$
\begin{aligned}
&\|\nabla \cdot A_t\|_{L^\infty} \le Ct^{1-\sigma},\\
&\|\phi_t - \phi\|_{L^\infty} \le Ct^{-1-\sigma},\\
&\|A_t - A\|_{L^\infty} \le Ct^{-\sigma},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla \cdot A_t\|_{L^2} \le Ct^{\frac{1}{2}-\sigma},\\
&\|\phi_t - \phi\|_{L^2} \le Ct^{-\frac{3}{2}-\sigma},\\
&\|A_t - A\|_{L^2} \le Ct^{-\frac{1}{2}-\sigma}.
\end{aligned}
$$

The following lemma is taken from [28].

Lemma 2.2 (Zhang)**.** *Let $\Omega \subset \mathbb{R}^n, n \ge 2$ be a bounded domain with C^2 boundary and $u \in H^1(\Omega)$. Then there exists a constant C such that*

$$
\int_{\partial\Omega} u^2 dS \le C\left\{\left(\int_\Omega u^2 dx\right)^{1/2}\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} + \int_\Omega u^2 dx\right\}.
$$

We will need the stable determination of the conductivity at points on the boundary of Ω. Since the stability estimate derived in [2] is local, the same estimates should hold for the localized Dirichlet-to-Neumann map. This result can be proved by the same arguments in [2].

Theorem 2.3. *Let $\gamma_j \in C^{1,\sigma}(\partial\Omega)$ for $j = 1, 2$. Then*

$$
\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_* \tag{2.1}
$$

and

$$
\sum_{|\alpha|=1} \|\partial^\alpha\gamma_1 - \partial^\alpha\gamma_2\|_{L^\infty(\partial\Omega)} \lesssim \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^\theta \tag{2.2}
$$

for some $0 < \theta < 1$ depending only on σ. Here the implicit constants depend on $n, \Omega, \sigma, \gamma_0$ and $\|\gamma_j\|_{C^{1,\sigma}(\overline{\Omega})}$ for $j = 1, 2$.

We will use the following theorem to obtain the stability estimate on a large ball $B(0, R)$ by controlling an open subset of $B(0, R)$. This idea was introduced in [13].

Proposition 2.4 (Vessella)**.** *Let $\tau_0, d_0 > 0$. Let $D \subset \mathbb{R}^n$ be an open, bounded and connected set such that $\{x \in D : d(x, \partial D) > \tau\}$ is connected for any $\tau \in [0, \tau_0]$. Let $E \subset D$ be an open set such that $d(E, \partial D) \ge d_0$. If f is an analytic function with*

$$
\|\partial^\alpha f\|_{L^\infty(D)} \le \frac{M\alpha!}{\rho^{|\alpha|}}, \quad \textit{for all } \alpha \in \mathbb{N}^n
$$

for some $M, \rho > 0$, then

$$
|f(x)| \le (2M)^{1-\tilde{\theta}(|E|/|D|)}(\|f\|_{L^\infty(E)})^{\tilde{\theta}(|E|/|D|)},
$$

where $\tilde{\theta} \in (0, 1)$ depends on d_0, $diam D$, τ_0, n, ρ and $d(x, \partial D)$.

3. Complex geometrical optics solutions

In this section, we will review the construction of CGO solutions for the conductivity equation following the arguments presented in [28], but with the conductivity in $C^{1,\sigma}, 0 < \sigma < 1$.

First, we introduce the spaces $\dot{X}^b_\zeta$ and X^b_ζ which are defined by the norm

$$\|u\|_{\dot{X}^b_\zeta} = \||p_\zeta(\xi)|^b \hat{u}(\xi)\|_{L^2}$$

and

$$\|u\|_{X^b_\zeta} = \|(|\zeta| + |p_\zeta(\xi)|)^b \hat{u}(\xi)\|_{L^2},$$

respectively. Here $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$ is the symbol of $\Delta + 2\zeta \cdot \nabla$.

Let Ω be an open bounded domain in $\mathbb{R}^n, n \geq 3$ with C^2 boundary. Let $\gamma \in C^{1,\sigma}(\overline{\Omega})$ and let u be the solution of $\nabla \cdot (\gamma \nabla u) = 0$ in Ω. Then u satisfies

$$(-\Delta - A \cdot \nabla)u = 0 \quad \text{in } \Omega, \tag{3.1}$$

where $A = \nabla \log \gamma \in C^{0,\sigma}(\overline{\Omega})$. Suppose that the CGO solutions of (3.1) are of the form

$$u = e^{-\frac{\phi_t}{2}} e^{x \cdot \zeta}(1 + w(x, \zeta)),$$

with $\phi_t = \Psi_t * \phi$ and $\zeta \in \mathbb{C}^n,\ \zeta \cdot \zeta = 0$. Here we denote $\phi = \log \gamma$. Then the function w satisfies the following equation

$$(-\Delta + (A_t - A) \cdot \nabla + q_t)(e^{x \cdot \zeta}(1 + w)) = 0, \tag{3.2}$$

where $q_t = \frac{1}{2}\nabla \cdot A_t - \frac{1}{4}(A_t)^2 + \frac{1}{2}A \cdot A_t$. Equivalently, w is the solution of

$$(-\Delta_\zeta + (A_t - A) \cdot \nabla_\zeta + q_t)w = (A - A_t) \cdot \zeta - q_t, \tag{3.3}$$

where $-\Delta_\zeta = \Delta + 2\zeta \cdot \nabla$ and $\nabla_\zeta = \nabla + \zeta$.

Given $k \in \mathbb{R}^n$. Let $\eta, \eta_1 \in S^{n-1}$ and $k, \eta, \eta_1 \in \mathbb{R}^n$ be mutually orthogonal. We choose $\zeta_1 = -s\eta - i(\frac{k}{2} - r\eta_1)$ and $\zeta_2 = s\eta - i(\frac{k}{2} + r\eta_1)$ such that $|k|^2/4 + r^2 = s^2$, $\zeta_i \cdot \zeta_i = 0$ and $\zeta_1 + \zeta_2 = -ik$.

Theorem 3.1 (Zhang)**.** *Let $\gamma \in C^1(\mathbb{R}^n)$ with $\gamma > \gamma_0 > 0$ and $\gamma = 1$ outside a ball. Then for any fixed $k \in \mathbb{R}^n$, there exists a sequence $\zeta_i^{(n)}$ with $|\zeta_i^{(n)}| = \sqrt{2}s_n$ such that*

$$\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}} \lesssim \|(A_{s_n} - A) \cdot \zeta_i^{(n)} + q_{s_n}\|_{\dot{X}^{-1/2}_{\zeta_i^{(n)}}} \to 0 \quad \text{as } s_n \to 0. \tag{3.4}$$

Moreover,

$$\|w_i^{(n)}\|_{L^2(\Omega)} \lesssim s_n^{-1/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}; \quad \|w_i^{(n)}\|_{H^1(\Omega)} \lesssim s_n^{1/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}$$

and

$$\|w_i^{(n)}\|_{H^{\frac{1}{2}}(\Omega)} \lesssim \|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}}; \quad \|w_i^{(n)}\|_{H^2(\Omega)} \lesssim s_n^{3/2}\|w_i^{(n)}\|_{\dot{X}^{1/2}_{\zeta_i^{(n)}}},$$

where $w_i^{(n)}$ is a solution of (3.3) with $t = s_n$ and $A = \nabla\phi_i = \nabla \log \gamma_i$.

From Theorem 3.1, we take the CGO solutions

$$u_1^{(n)} = e^{-\frac{\phi_{1s_n}}{2}} e^{x \cdot \zeta_1^{(n)}}(1 + w_1^{(n)})$$

and

$$u_2^{(n)} = e^{-\frac{\phi_{2s_n}}{2}} e^{x \cdot \zeta_2^{(n)}}(1 + w_2^{(n)}).$$

The CGO solutions can also be written as

$$u_i^{(n)} = e^{-\frac{\phi_{is_n}}{2}} e^{x \cdot \zeta_i^{(n)}}(1 + w_i^{(n)}) = \sqrt{\gamma_i}^{-1} e^{x \cdot \zeta_i^{(n)}}(1 + \psi_i^{(n)}) \tag{3.5}$$

for $i = 1, 2$. Here $\psi_i^{(n)} = \sqrt{\gamma_i}(e^{-\frac{\phi_{is_n}}{2}} - \sqrt{\gamma_i}^{-1}) + \sqrt{\gamma_i}e^{-\frac{\phi_{is_n}}{2}} w_i^{(n)}$. For simplicity, we will not write the superscripts (n) and the subscripts of s_n unless otherwise particularly specified.

Note that by lemma 2.1 and Theorem 3.1, we have

$$\|\psi_i\|_{L^2(\Omega)} \lesssim s^{-1-\sigma} + s^{-1/2}\|w_i\|_{\dot{X}^{1/2}_{\zeta_i}}. \tag{3.6}$$

Lemma 3.2. *For $0<\sigma<1$, if λ is sufficiently large we have*

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|(A_s-A)\cdot\zeta+q_s\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta \lesssim \lambda^{-2\sigma}+\lambda^{-1}. \tag{3.7}$$

Proof. Let Φ be a cut-off function on the support of A_s and A. Then, by Lemma 2.2 in [9], we have

$$\|(A_s)^2\|^2_{\dot{X}^{-1/2}_{\zeta}} = \|\Phi(A_s)^2\|^2_{\dot{X}^{-1/2}_{\zeta}} \lesssim \|(A_s)^2\|^2_{X^{-1/2}_{\zeta}} \lesssim s^{-1},$$
$$\|A\cdot A_s\|^2_{\dot{X}^{-1/2}_{\zeta}} = \|\Phi(A\cdot A_s)\|^2_{\dot{X}^{-1/2}_{\zeta}} \lesssim \|A\cdot A_s\|^2_{X^{-1/2}_{\zeta}} \lesssim s^{-1}.$$

Observing that $|(\nabla\cdot A_s)\hat{}(\xi)| = |\xi\cdot\hat{A}_s| = |\xi\cdot\hat{\Psi}(\frac{\xi}{s})\hat{A}(\xi)| \le \|\hat{\Psi}(\frac{\xi}{s})\|_{L^\infty}|\xi\cdot\hat{A}| \lesssim |(\nabla\cdot A)\hat{}(\xi)|$. Let $h=\sqrt{\lambda}$ and $\Psi_h = h^n\Psi(hx)$ as in Lemma 2.1. Using Lemma 3.1 in [9], we have

$$\begin{aligned}
&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A_s\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot(\Psi_h*A)\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta+\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot(\Psi_h*A-A)\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\frac{1}{\lambda}\|\nabla\cdot(\Psi_h*A)\|^2_{L^2}+\|\Psi_h*A-A\|^2_{L^2}\\
\lesssim&\frac{1}{\lambda}h^{1-2\sigma}+h^{-1-2\sigma}\\
\lesssim&\lambda^{-\frac{1}{2}-\sigma}.
\end{aligned} \tag{3.8}$$

By the definition of q_s, we can deduce that

$$\begin{aligned}
&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|q_s\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|\nabla\cdot A_s\|^2_{\dot{X}^{-1/2}_{\zeta}}+\|(A_s)^2\|^2_{\dot{X}^{-1/2}_{\zeta}}+\|A\cdot A_s\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\lambda^{-\frac{1}{2}-\sigma}+\lambda^{-1}.
\end{aligned} \tag{3.9}$$

Using Lemma 2.1, we get that

$$\begin{aligned}
&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}\|(A_s-A)\cdot\zeta\|^2_{\dot{X}^{-1/2}_{\zeta}}dsd\eta\\
\lesssim&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}s\|A_s-A\|^2_{L^2}dsd\eta\\
\lesssim&\frac{1}{\lambda}\int_{S^{n-1}}\int_{\lambda}^{2\lambda}s^{-2\sigma}dsd\eta\\
\lesssim&\lambda^{-2\sigma}.
\end{aligned} \tag{3.10}$$

□

Note that $\|w\|^2_{\dot{X}^{1/2}_\zeta} \lesssim \|(A_s - A)\cdot\zeta + q_s\|^2_{\dot{X}^{-1/2}_\zeta}$. By lemma 3.2, we obtain the following estimate

$$\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda}\|w\|^2_{\dot{X}^{1/2}_\zeta}dsd\eta \lesssim \lambda^{-2\sigma}+\lambda^{-1}. \tag{3.11}$$

The following Carleman estimate is deduced by Zhang by using the Carleman estimate in the paper [16].

Theorem 3.3 (Zhang)**.** *Let $\eta \in S^{n-1}$ and $u \in H^2(\Omega)$. Suppose that $\gamma \in C^1(\Omega)$. Then there exists a constant $s_0 > 0$ such that for $s \geq s_0$, we have*

$$\begin{aligned}
&C(s^2\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}) - C_1 s^2\int_{\partial\Omega}|u|^2 dS\\
&- C_2\int_{\partial\Omega}\overline{u}\partial_\nu u dS + \int_{\partial\Omega}4s\Re(\partial_\nu u\partial_\eta\overline{u}) - 2s(\nu\cdot\eta)|\nabla u|^2 + 2s^3(\nu\cdot\eta)|u|^2 dS\\
&\leq \|e^{-x\cdot s\eta}(-\Delta + (A_s - A)\cdot\nabla + q_s)(e^{x\cdot s\eta}u)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.12}$$

Proposition 3.4 (Knudsen)**.** *Suppose $\gamma_j \in C^1(\overline{\Omega})$, $j = 1, 2$ and suppose that $u_1, u_2 \in H^1(\overline{\Omega})$ satisfy $\nabla\cdot\gamma_j\nabla u_j = 0$ in Ω. Suppose that $\tilde{u}_1 \in H^1(\overline{\Omega})$ satisfies $\nabla\cdot\gamma_1\nabla\tilde{u}_1 = 0$ with $\tilde{u}_1 = u_2$ on $\partial\Omega$. Then*

$$\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla(u_1u_2)dx = \int_{\partial\Omega}\gamma_1\partial_\nu(\tilde{u}_1 - u_2)u_1 dS, \tag{3.13}$$

where the integral is understood in the sense of the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

Proposition 3.4 implies that

$$\begin{aligned}
&\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2} - \sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla(u_1u_2)dx\\
&= \int_{\partial\Omega_{+,\varepsilon}}\gamma_1\partial_\nu(\tilde{u}_1 - u_2)u_1 dS + \int_{\partial\Omega_{-,\varepsilon}}\gamma_1\partial_\nu(\tilde{u}_1 - u_2)u_1 dS.
\end{aligned} \tag{3.14}$$

In the rest part of this section, we will estimate the right hand side of (3.14). First, we estimate the second term on the right hand side of (3.14). Using Theorem 2.3 and trace theorem, we have

$$\begin{aligned}
&|\int_{\partial\Omega_{-,\varepsilon}}\gamma_1\partial_\nu(\tilde{u}_1 - u_2)u_1 dS|^2\\
\lesssim&\|1+w_1\|^2_{L^2(\partial\Omega_{-,\varepsilon})}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\gamma_1\partial_\nu(\tilde{u}_1 - u_2)|^2 dS\\
\lesssim&\|1+w_1\|^2_{L^2(\partial\Omega_{-,\varepsilon})}\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}(|\gamma_1\partial_\nu\tilde{u}_1 - \gamma_2\partial_\nu u_2|^2 + |(\gamma_1-\gamma_2)\partial_\nu u_2|^2)dS\\
\lesssim&\|1+w_1\|^2_{L^2(\partial\Omega_{-,\varepsilon})}e^{cs}(\|(\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2})u_2\|^2_{L^2(\partial\Omega_{-,\varepsilon})} + \|\gamma_1-\gamma_2\|^2_{L^\infty(\partial\Omega)}\|\tilde{\Lambda}_{\gamma_2}u_2\|^2_{L^2(\partial\Omega_{-,\varepsilon})})\\
\lesssim&\|1+w_1\|^2_{L^2(\partial\Omega_{-,\varepsilon})}e^{cs}(\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)} + \|\gamma_1-\gamma_2\|^2_{L^\infty(\partial\Omega)}\|\tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)})\\
\lesssim&e^{cs}\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)}.
\end{aligned} \tag{3.15}$$

Note that by Lemma 2.2 and Lemma 3.1, we have

$$\|1+w_1\|^2_{L^2(\partial\Omega_{-,\varepsilon})} \lesssim 1 + \|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}},$$

where $\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}}$ is neglected if s is sufficiently large.

Denote $u_0 = e^{\frac{\phi_{1s}}{2}}(\tilde{u}_1 - u_2)$ and $\delta u = (e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}})u_2$. Let $u = u_0 + \delta u$. The first term on the right hand side of (3.14) is bounded by

$$\begin{aligned}
&|\int_{\partial\Omega_{+,\varepsilon}} \gamma_1 \partial_\nu(\tilde{u}_1 - u_2)u_1 dS|^2 \\
\lesssim &\|1 + w_1\|^2_{L^2(\partial\Omega_{+,\varepsilon})} \int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu(\tilde{u}_1 - u_2)|^2 dS \\
\lesssim &\Big(1 + \|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}\Big)\Big(\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2 dS + \int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu \delta u|^2 dS\Big).
\end{aligned} \tag{3.16}$$

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be an open, bounded domain with C^2 boundary. For $i = 1, 2$, let $\gamma_i \in C^{1,\sigma}(\overline{\Omega}) \cap H^{\frac{3}{2}+\sigma}(\Omega)$ be a real-valued function and $\gamma_i > \gamma_0 > 0$. Suppose that $\gamma_1|_{\partial\Omega_{+,\varepsilon}} = \gamma_2|_{\partial\Omega_{+,\varepsilon}}$ and $\partial_\nu\gamma_1|_{\partial\Omega_{+,\varepsilon}} = \partial_\nu\gamma_2|_{\partial\Omega_{+,\varepsilon}}$. Then*

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS \lesssim s^{-2\sigma}(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}), \tag{3.17}$$

$$\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS \lesssim s^{-2-2\sigma}(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}) \tag{3.18}$$

and

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS \lesssim (s^{-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_* + s^2\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}), \tag{3.19}$$

$$\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS \lesssim (s^{-2-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}) \tag{3.20}$$

when s is sufficiently large.

Proof. We will prove the estimate for $\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS$ first.

$$\begin{aligned}
&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS \\
\lesssim &\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla(e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}})|^2|u_2|^2 dS + \int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}}|^2|\nabla u_2|^2 dS.
\end{aligned} \tag{3.21}$$

Using Theorem 2.3, the first term of the right side of (3.21) can be written as

$$\begin{aligned}
&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla(e^{\frac{\phi_{1s}}{2}} - e^{\frac{\phi_{2s}}{2}})|^2|u_2|^2 dS \\
\lesssim &\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}(|\nabla(e^{\frac{\phi_{1s}}{2}} - \sqrt{\gamma_1})|^2 + |\nabla(\sqrt{\gamma_1} - \sqrt{\gamma_2})|^2 + |\nabla(e^{\frac{\phi_{2s}}{2}} - \sqrt{\gamma_2})|^2)|u_2|^2 dS \\
\lesssim &\sum_{j=1}^{2}(\|A_{js} - A_j\|^2_{L^\infty(\overline{\Omega})} + \|\nabla(\gamma_1 - \gamma_2)\|^2_{L^\infty(\partial\Omega)} + \|\gamma_1 - \gamma_2\|^2_{L^\infty(\partial\Omega)} \\
&+ \|\phi_{js} - \phi_j\|^2_{L^\infty(\overline{\Omega})})(\|1 + w_2\|^2_{L^2(\partial\Omega)}) \\
\lesssim &(s^{-2\sigma} + s^{-2-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_* + +\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}).
\end{aligned}$$

To estimate the second term of (3.21),

$$
\begin{aligned}
&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|e^{\frac{\phi_{1s}}{2}}-e^{\frac{\phi_{2s}}{2}}|^2|\nabla u_2|^2 dS\\
\lesssim&\int_{\partial\Omega_{-,\varepsilon}}(|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}|^2+|\sqrt{\gamma_1}-\sqrt{\gamma_2}|^2+|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}|^2)|\nabla w_2|^2 dS\\
&+s^2\int_{\partial\Omega_{-,\varepsilon}}(|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}|^2+|\sqrt{\gamma_1}-\sqrt{\gamma_2}|^2+|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}|^2)(1+|w_2|^2)dS\\
\lesssim&\sum_{j=1}^{2}(\|\phi_{js}-\phi_j\|^2_{L^\infty(\overline{\Omega})}+\|\sqrt{\gamma_1}-\sqrt{\gamma_2}\|^2_{L^\infty(\partial\Omega)}\Big(\|\nabla w_2\|^2_{L^2(\partial\Omega)}+s^2(1+\|w_2\|^2_{L^2(\partial\Omega)})\Big)\\
\lesssim&(s^{-2\sigma}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2)(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}).
\end{aligned}
$$

Thus we have

$$
\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS\lesssim(s^{-2\sigma}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2)(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}).
$$

Since $\gamma_1|_{\partial\Omega_{+,\varepsilon}}=\gamma_2|_{\partial\Omega_{+,\varepsilon}}$ and $\partial_\nu\gamma_1|_{\partial\Omega_{+,\varepsilon}}=\partial_\nu\gamma_2|_{\partial\Omega_{+,\varepsilon}}$, the estimate of $\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\nabla\delta u|^2dS$ does not contain the $\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*$ terms. Thus,

$$
\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\nabla\delta u|^2 dS\lesssim s^{-2\sigma}(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}).
$$

Similarly, we can deduce that

$$
\begin{aligned}
&\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS\\
\lesssim&\int_{\partial\Omega_{-,\varepsilon}}(|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}|^2+|\sqrt{\gamma_1}-\sqrt{\gamma_2}|^2+|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}|^2)(1+|w_2|^2)dS\\
\lesssim&(s^{-2-2\sigma}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2)(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\delta u|^2 dS\\
\lesssim&\int_{\partial\Omega_{+,\varepsilon}}(|e^{\frac{\phi_{1s}}{2}}-\sqrt{\gamma_1}|^2+|e^{\frac{\phi_{2s}}{2}}-\sqrt{\gamma_2}|^2)(1+|w_2|^2)dS\\
\lesssim&s^{-2-2\sigma}(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}).
\end{aligned}
$$

□

Lemma 3.6. *Under the same assumption as Lemma 3.5, we have*

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2 dS\\
\lesssim&s^{-2\sigma}+s^{-1}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\\
&+e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\|u_2\|^2_{H^1(\Omega)}.
\end{aligned}
\tag{3.22}
$$

for some $0<\theta<1$ when s is sufficiently large.

Proof. Note that since $u_0|_{\partial\Omega}=0$, we have

$$
\begin{aligned}
|\int_{\partial\Omega_{-,\varepsilon}}(\nu\cdot\eta)e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS|\lesssim&\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu(\tilde u_1-u_2)|^2 dS\\
\lesssim&e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\|u_2\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Let $v = e^{-x\cdot s\eta}u$. We plug v into the Carleman estimate in Theorem 3.3, then we get that

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} 4\Re(\partial_\nu v \partial_\eta \overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2 dS \\
\lesssim & s\int_{\partial\Omega} |v|^2 dS + \int_{\partial\Omega} s^2(\nu\cdot\eta)|v|^2 dS + \frac{1}{s}\int_{\partial\Omega} \overline{v}\partial_\nu v dS \\
&+ \frac{1}{s}\|e^{-x\cdot s\eta}(-\Delta + (A_{1s} - A)\cdot\nabla + q_{1s})(e^{x\cdot s\eta}v)\|^2_{L^2(\Omega)} \\
&+ \int_{\partial\Omega_{-,\varepsilon}} 4\Re(\partial_\nu v\partial_\eta\overline{v}) - 2(\nu\cdot\eta)|\nabla v|^2 dS \\
=: & I + II + III + IV + V.
\end{aligned}
$$

For I and II, since $u_0|_{\partial\Omega} = 0$, it follows that

$$
\begin{aligned}
s\int_{\partial\Omega}|v|^2 dS \lesssim & s\int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS \\
\lesssim & (s^{-1-2\sigma} + s\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}})
\end{aligned}
$$

and

$$
\begin{aligned}
s^2\int_{\partial\Omega}(\nu\cdot\eta)|v|^2 dS \lesssim & s^2\int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS \\
\lesssim & (s^{-2\sigma} + s^2\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}).
\end{aligned}
$$

To estimate III,

$$
\begin{aligned}
&\frac{1}{s}\int_{\partial\Omega}\overline{v}\partial_\nu v dS \\
=&\frac{1}{s}\int_{\partial\Omega} e^{-x\cdot s\eta}\overline{\delta u}\partial_\nu(e^{-x\cdot s\eta}u) dS \\
=&-(\nu\cdot\eta)\int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS + \frac{1}{s}\int_{\partial\Omega} e^{-2x\cdot s\eta}\overline{\delta u}\partial_\nu u dS \\
\lesssim & \int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS + \frac{1}{s}\int_{\partial\Omega} e^{-2x\cdot s\eta}|\delta u|^2 dS + \frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu \delta u|^2 dS \\
&+ \frac{1}{s}\int_{\partial\Omega_{-,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u_0|^2 dS + \frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2 dS \\
\lesssim & s^{-1}(s^{-2\sigma} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^{2\theta}_* + s^2\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*)(1 + \|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}) \\
&+ \frac{1}{s}e^{cs}\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)} + \frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2 dS.
\end{aligned}
$$

Using Lemma 2.1 to estimate IV.

$$
\begin{aligned}
&\frac{1}{s}\|e^{-x\cdot s\eta}(-\Delta+(A_{1s}-A)\cdot\nabla+q_{1s})(e^{x\cdot s\eta}v)\|^2_{L^2(\Omega)}\\
\leq&\frac{1}{s}\int_\Omega e^{-2x\cdot s\eta}|(-\Delta+(A_{1s}-A)\cdot\nabla+q_{1s})e^{x\cdot\zeta_2}(1+w_2)|^2dx\\
\leq&\frac{1}{s}\int_\Omega e^{-2x\cdot s\eta}|((A_{1s}-A)-(A_{2s}-A_2))\cdot\nabla(e^{x\cdot\zeta_2}(1+w_2))+(q_{1s}-q_{2s})e^{x\cdot\zeta_2}(1+w_2)|^2dx\\
\lesssim&s\int_\Omega|A_2-A_{2s}|^2+|A_1-A_{1s}|^2dx+s\int_\Omega(|A_2-A_{2s}|^2+|A_1-A_{1s}|^2)|w_2|^2dx\\
&+\frac{1}{s}\int_\Omega(|A_2-A_{2s}|^2+|A_1-A_{1s}|^2)|\nabla w_2|^2dx+\frac{1}{s}\int_\Omega|q_{2s}-q_{1s}|^2dx\\
&+\frac{1}{s}\int_\Omega|q_{2s}-q_{1s}|^2|w_2|^2dx\\
\lesssim&s\sum_{j=1}^2\|A_{js}-A_j\|^2_{L^2}+\sum_{j=1}^2\|A_{js}-A_j\|^2_{L^\infty}\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}+\frac{1}{s}\|q_{2s}-q_{1s}\|^2_{L^2}\\
&+\frac{1}{s^2}\|q_{2s}-q_{1s}\|^2_{L^\infty}\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}\\
\lesssim&s^{-2\sigma}+s^{-1}+s^{-2\sigma}\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}.
\end{aligned}
$$

Now, for V, since $u_0|_{\partial\Omega}=0$ implies that $\nabla u_0=\partial_\nu u_0$ on $\partial\Omega$. Then we have

$$
\begin{aligned}
&|\int_{\partial\Omega_{-,\varepsilon}}4\Re(\partial_\nu v\partial_\eta\overline{v})-2(\nu\cdot\eta)|\nabla v|^2dS|\\
\lesssim&\int_{\partial\Omega_{-,\varepsilon}}|\nabla v|^2dS\\
\lesssim&s^2\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\delta u|^2dS+\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\nabla\delta u|^2dS+\int_{\partial\Omega_{-,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u_0|^2dS\\
\lesssim&(s^{-2\sigma}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^{2\theta}_*+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^2_*)(1+\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}})\\
&+e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Combing the estimates from I to V, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}}4\Re(\partial_\nu v\partial_\eta\overline{v})-2(\nu\cdot\eta)|\nabla v|^2dS\\
\lesssim&s^{-2\sigma}+s^{-1}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^{2\theta}_*+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^2_*\\
&+e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|^2_*\|u_2\|^2_{H^1(\Omega)}+\frac{1}{s}\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\partial_\nu u|^2dS
\end{aligned}
\tag{3.23}
$$

since $\|w_2\|^2_{\dot X^{1/2}_{\zeta_2}}$ can be neglected when s is sufficiently large.

Moreover, for $(\nu\cdot\eta)>\varepsilon>0$, we have

$$
\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}}4\Re(\partial_\nu v\partial_\eta\overline{v})-2(\nu\cdot\eta)|\nabla v|^2dS\\
\geq&\int_{\partial\Omega_{+,\varepsilon}}(\nu\cdot\eta)e^{-2x\cdot s\eta}|\partial_\nu u|^2dS-s^2\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\delta u|^2dS-\int_{\partial\Omega_{+,\varepsilon}}e^{-2x\cdot s\eta}|\nabla\delta u|^2.
\end{aligned}
\tag{3.24}
$$

Combining (3.23) and (3.24), we can deduce that

$$\begin{aligned}
&\int_{\partial\Omega_{+,\varepsilon}} e^{-2x\cdot s\eta}|\partial_\nu u|^2 dS \\
\lesssim & s^{-2\sigma} + s^{-1} + \|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^{2\theta} + s^2\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2 \\
& + e^{cs}\|\tilde{\Lambda}_{\gamma_1} - \tilde{\Lambda}_{\gamma_2}\|_*^2\|u_2\|_{H^1(\Omega)}^2.
\end{aligned}$$

if s is sufficiently large. □

4. Stability result

We consider the function $v := \log\sqrt{\gamma_1} - \log\sqrt{\gamma_2} \in H^1(\Omega)$. This function v is a weak solution of

$$\Delta v + \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})\nabla v = F \quad \text{in } \Omega \qquad v|_{\partial\Omega} = (\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2})|_{\partial\Omega}, \tag{4.1}$$

with $F \in H^{-1}(\Omega)$.

Recall that in section 2, we extend γ_j to be the function in $\mathbb{R}^n$ such that $\gamma_j \in C^{1,\sigma}(\mathbb{R}^n)$ and $\gamma_j - 1 \in H^{\frac{3}{2}+\sigma}(\mathbb{R}^n)$ with $\mathrm{supp}(\gamma_j - 1) \subset \overline{B}$. Hence v is also a weak solution of the elliptic equation $\nabla\cdot(\sqrt{\gamma_1}\sqrt{\gamma_2})\nabla v = (\sqrt{\gamma_1}\sqrt{\gamma_2})\cdot F$ in B, we get the following estimate

$$\|v\|_{H^1(B)} \lesssim \|F\|_{H^{-1}(B)} \lesssim \|F\|_{H^{-1}(\mathbb{R}^n)}. \tag{4.2}$$

The goal now is to bound $\|F\|_{H^{-1}(\mathbb{R}^n)}$. Following the argument in [10] and (4.1), let $g = \nabla(\log\sqrt{\gamma_1} + \log\sqrt{\gamma_2})$ and denote by $\tilde{f}$ the extension of $f \in L^2(\Omega)$ by zero to $\mathbb{R}^n$. Then for $\varphi \in H_0^1(\Omega)$ we have

$$\begin{aligned}
\langle F, \varphi\rangle &= \int_\Omega -\nabla v\nabla\overline{\varphi} + (g\nabla v)\overline{\varphi}dx \\
&= \int_{\mathbb{R}^n} -\widetilde{\nabla v}\nabla\overline{\tilde{\varphi}} + (g\widetilde{\nabla v})\overline{\tilde{\varphi}}dx \\
&= \int_{\mathbb{R}^n} \left((ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\right)\overline{\mathcal{F}\tilde{\varphi}}dk.
\end{aligned}$$

Hence

$$|\langle F, \varphi\rangle| = \left(\int_{\mathbb{R}^n} |(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})|^2(1+|k|^2)^{-1}dk\right)^{\frac{1}{2}}\|\varphi\|_{H^1(\mathbb{R}^n)}.$$

Here $\mathcal{F}$ denotes the Fourier transform. Since $\gamma_i \in H^{\frac{3}{2}+\sigma}(\Omega)$, it follows that

$$\begin{aligned}
\|F\|_{H^{-1}(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} |(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})|^2(1+|k|^2)^{-1}dk \\
&\le \int_{|k|\le R} |(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})|^2(1+|k|^2)^{-1}dk \\
&\quad + \int_{|k|>R} |(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})|^2(1+|k|^2)^{-1}dk \\
&\lesssim R^n\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|_{L^\infty(B(0,R))}^2 \\
&\quad + \frac{1}{R^2}\|g\widetilde{\nabla v}\|_{L^2(\mathbb{R}^n)} + \int_{|k|>R}(1+|k|^2)^{\frac{1}{2}}|\mathcal{F}\widetilde{\nabla v}|^2(1+|k|^2)^{-\frac{1}{2}}dk \\
&\lesssim R^n\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|_{L^\infty(B(0,R))}^2 \\
&\quad + \frac{1}{R^2}\|g\widetilde{\nabla v}\|_{L^2(\mathbb{R}^n)} + \frac{1}{R}\|\nabla v\|_{H^{\frac{1}{2}}(\Omega)}^2.
\end{aligned} \tag{4.3}$$

Now we will need to estimate $\|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|_{L^\infty(B(0,R))}^2$.

From (3.14), (3.15), (3.16) and Lemma 3.6, we have

$$
\begin{aligned}
&|\int_\Omega (\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla(u_1u_2)dx|^2\\
\leq &|\int_{\partial\Omega_{+,\varepsilon}}\gamma_1\partial_\nu(\tilde u_1-u_2)u_1dS|^2+|\int_{\partial\Omega_{-,\varepsilon}}\gamma_1\partial_\nu(\tilde u_1-u_2)u_1dS|^2\\
\lesssim &s^{-2\sigma}+s^{-1}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\\
&+e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\|u_2\|_{H^1(\Omega)}^2.
\end{aligned}\tag{4.4}
$$

Denote $q=(ik)\widetilde{\nabla v}+(g\widetilde{\nabla v})$. Plug $u_i=\sqrt{\gamma_i}^{-1}e^{x\cdot\zeta_i}(1+\psi_i), i=1,2$, into (4.4), we obtain that

$$
\begin{aligned}
|\mathcal{F}(q)(k)|^2=&|\int_\Omega e^{-ik\cdot x}(ik\nabla(\log\sqrt{\gamma_1}-\log\sqrt{\gamma_2})+(\nabla\log\sqrt{\gamma_1})^2-(\nabla\log\sqrt{\gamma_2})^2)dx|^2\\
\leq&|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla\Big(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\Big)dx|^2\\
&+s^{-2\sigma}+s^{-1}+\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta}+s^2\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2+e^{cs}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^2\|u_2\|_{H^1(\Omega)}^2.
\end{aligned}\tag{4.5}
$$

To estimate

$$
\begin{aligned}
&|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla\Big(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}(\psi_1+\psi_2+\psi_1\psi_2)\Big)dx|^2\\
\lesssim&|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot\nabla\Big(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x}\Big)(\psi_1+\psi_2+\psi_1\psi_2)dx|^2\\
&+|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x})(\nabla\psi_1+\nabla\psi_2+\nabla(\psi_1\psi_2))dx|^2\\
=:&I+II.
\end{aligned}
$$

For I, using Theorem 3.1 and the definition of $\psi_i=\sqrt{\gamma_i}(e^{-\frac{\phi_{is}}{2}}-\sqrt{\gamma_i}^{-1})+\sqrt{\gamma_i}e^{-\frac{\phi_{is}}{2}}w_i=:\psi^{i1}+\psi^{i2}$, we can deduce that

$$
\begin{aligned}
I\lesssim&(|k|^2+1)(\|\psi_1\|_{L^2(\Omega)}^2+\|\psi_2\|_{L^2(\Omega)}^2+\|\psi_1\|_{L^2(\Omega)}^2\|\psi_2\|_{L^2(\Omega)}^2)\\
\lesssim&(|k|^2+1)(s^{-2-2\sigma}+s^{-1}(\|w_1\|_{\dot X_{\zeta_1}^{1/2}}^2+\|w_2\|_{\dot X_{\zeta_2}^{1/2}}^2))\\
\lesssim&|k|^2(s^{-2-2\sigma}+s^{-1}(\|w_1\|_{\dot X_{\zeta_1}^{1/2}}^2+\|w_2\|_{\dot X_{\zeta_2}^{1/2}}^2)).
\end{aligned}
$$

To estimate II, we divide it into two parts.

$$
\begin{aligned}
II\lesssim&|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x})(\nabla\psi^{11}+\nabla\psi^{21}+\nabla(\psi_1\psi_2))dx|^2\\
&+|\int_\Omega(\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x})(\nabla\psi^{12}+\nabla\psi^{22})dx|^2\\
=:&J+K.
\end{aligned}
$$

For J, using Lemma 2.1,

$$
\begin{aligned}
J\lesssim&s^{-2\sigma}+\|\psi_1\|_{L^2}\|\nabla\psi_2\|_{L^2}+\|\psi_2\|_{L^2}\|\nabla\psi_1\|_{L^2}\\
\lesssim&s^{-2\sigma}+(s^{-1-\sigma}+s^{-\frac12}\|w_1\|_{\dot X_{\zeta_1}^{1/2}})(s^{-\sigma}+s^{\frac12}\|w_2\|_{\dot X_{\zeta_2}^{1/2}})\\
&+(s^{-1-\sigma}+s^{-\frac12}\|w_2\|_{\dot X_{\zeta_2}^{1/2}})(s^{-\sigma}+s^{\frac12}\|w_1\|_{\dot X_{\zeta_1}^{1/2}}).
\end{aligned}
$$

To estimate K, we deduce that

$$
\begin{aligned}
&|\int_\Omega (\sqrt{\gamma_1}\nabla\sqrt{\gamma_2}-\sqrt{\gamma_2}\nabla\sqrt{\gamma_1})\cdot(\frac{1}{\sqrt{\gamma_1}\sqrt{\gamma_2}}e^{-ik\cdot x})(\nabla\psi^{12}+\nabla\psi^{22})dx|^2\\
\lesssim&\|w_1\|^2_{L^2(\Omega)}+\|w_2\|^2_{L^2(\Omega)}+\|\Phi_B w_1\|_{H^{\frac12}(\mathbb{R}^n)}+\|\Phi_B w_2\|_{H^{\frac12}(\mathbb{R}^n)}\\
\lesssim&s^{-1}(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}})+(\|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}})
\end{aligned}
$$

by using Theorem 3.1. Here Φ_B is some Schwartz function with compact support.

Based on the argument above, we have the following estimate.

$$
\begin{aligned}
|\mathcal{F}(q)(k)|^2\lesssim&|k|^2(s^{-2-2\sigma}+s^{-1}(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}))+(\|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}})\\
&+s^{-2\sigma}+s^{-1}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{cs}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2.
\end{aligned}
\tag{4.6}
$$

Note that $\|u_2\|^2_{H^1(\Omega)}\lesssim e^{cs}$.

Integrating on both side of (4.6) and using estimate (3.11), we get

$$
\begin{aligned}
|\mathcal{F}(q)(k)|^2\lesssim&|k|^2(\lambda^{-2-2\sigma}+\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda}s^{-1}(\|w_1\|^2_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|^2_{\dot{X}^{1/2}_{\zeta_2}}))dsd\eta\\
&+\frac{1}{\lambda}\int_{S^{n-1}}\int_\lambda^{2\lambda}(\|w_1\|_{\dot{X}^{1/2}_{\zeta_1}}+\|w_2\|_{\dot{X}^{1/2}_{\zeta_2}})dsd\eta\\
&+\lambda^{-2\sigma}+\lambda^{-1}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\\
\lesssim&|k|^2\left(\lambda^{-2-2\sigma}+\lambda^{-1}(\lambda^{-2\sigma}+\lambda^{-1})\right)+(\lambda^{-2\sigma}+\lambda^{-1})\\
&+\lambda^{-2\sigma}+\lambda^{-1}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\\
\lesssim&|k|^2(\lambda^{-1-2\sigma}+\lambda^{-2})+\lambda^{-2\sigma}+\lambda^{-1}\\
&+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2.
\end{aligned}
\tag{4.7}
$$

Varying η in a small conic neighborhood $U_\eta\in S^{n-1}$, we get the estimate (4.7) uniformly for all $k\in E=\{k\in\mathbb{R}^n: k \text{ orthogonl to some } \tilde{\eta}\in U_\eta\}$.

Fixed $R>0$ and $k\in\mathbb{R}^n$. Let $f(k)=\mathcal{F}(q)(Rk)$. Since q is compactly supported, $\mathcal{F}(q)$ is analytic by the Paley-Wiener theorem and

$$
|D^\alpha f(k)|\le\|q\|_{L^1(\Omega)}\frac{R^{|\alpha|}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}\le C\frac{R^{|\alpha|}}{\alpha!(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}\alpha!\le C\frac{e^{nR}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}\alpha!
$$

for any $\alpha\in\mathbb{N}^n$. Let $D=B(0,2)$ and $\tilde{E}=E\cap B(0,1)$ with $M=Ce^{nR}$ and $\rho=\mathrm{diam}(\Omega)^{-1}$. From Proposition 2.4, there exists $\tilde{\theta}\in(0,1)$ such that

$$
|\mathcal{F}q(k)|=|f(k/R)|\le Ce^{nR(1-\tilde{\theta})}\|f\|^{\tilde{\theta}}_{L^\infty(\tilde{E})}\le Ce^{nR(1-\tilde{\theta})}\|\mathcal{F}q(k)\|^{\tilde{\theta}}_{L^\infty(E)}
\tag{4.8}
$$

for all $k\in B(0,R)$.

Using (4.8), together with (4.7) and (4.3), we get

$$
\begin{aligned}
\|F\|^2_{H^{-1}(\mathbb{R}^n)}\lesssim&R^ne^{2nR(1-\tilde{\theta})}\Big(R^2(\lambda^{-1-2\sigma}+\lambda^{-2})+\lambda^{-2\sigma}+\lambda^{-1}\\
&+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\Big)^{\tilde{\theta}}+R^{-1}\\
\lesssim&R^ne^{2nR(1-\tilde{\theta})}\Big(\lambda^{-2\sigma}+\lambda^{-1}+\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^{2\theta}+e^{c\lambda}\|\tilde{\Lambda}_{\gamma_1}-\tilde{\Lambda}_{\gamma_2}\|_*^2\Big)^{\tilde{\theta}}+R^{-1}
\end{aligned}
$$

if $\lambda > R^2 > 1$. Thus,

$$\begin{aligned}\|F\|^{\frac{2}{\tilde\theta}}_{H^{-1}(\mathbb{R}^n)} \lesssim & R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}}(\lambda^{-2\sigma}+\lambda^{-1}) + R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} \|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta} \\ & + R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} e^{c\lambda}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2} + R^{\frac{-1}{\tilde\theta}} \\ \lesssim & R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}}\lambda^{-2\beta} + R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} \|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta} \\ & + R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} e^{c\lambda}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2} + R^{\frac{-1}{\tilde\theta}}.\end{aligned} \tag{4.9}$$

Here

$$\begin{cases} \beta = \sigma & \text{if } 0<\sigma\le \frac12, \\ \beta = \frac12 & \text{if } \frac12 < \sigma < 1 \; . \end{cases}$$

Choosing

$$\lambda = (R^{n+1}e^{2nR(1-\tilde\theta)})^{\frac{1}{2\beta\tilde\theta}}$$

such that

$$R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}}\lambda^{-2\beta} = R^{\frac{-1}{\tilde\theta}},$$

we proceed

$$\|F\|^{\frac{2}{\tilde\theta}}_{H^{-1}(\mathbb{R}^n)} \lesssim R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} \|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta} + R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}} e^{c\lambda}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2} + R^{\frac{-1}{\tilde\theta}}. \tag{4.10}$$

Using the fact that

$$\begin{aligned} R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}+c\lambda} &= R^{\frac{n}{\tilde\theta}} e^{2nR\frac{1-\tilde\theta}{\tilde\theta}+c(R^{n+1}e^{2nR(1-\tilde\theta)})^{\frac{1}{2\beta\tilde\theta}}} \\ &\le \exp\left(e^{[\frac{n}{\tilde\theta}+2n\frac{1-\tilde\theta}{\tilde\theta}+\frac{n+1}{2\beta\tilde\theta}+\frac{n(1-\tilde\theta)}{\beta\tilde\theta}]R}\right) \quad \text{for all } R>0. \end{aligned}$$

Setting $K = \frac{n}{\tilde\theta}+2n\frac{1-\tilde\theta}{\tilde\theta}+\frac{n+1}{2\beta\tilde\theta}+\frac{n(1-\tilde\theta)}{\beta\tilde\theta}$, (4.10) leads to

$$\|F\|^{\frac{2}{\tilde\theta}}_{H^{-1}(\mathbb{R}^n)} \lesssim e^{e^{KR}}(\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2\theta} + \|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{2}) + R^{\frac{-1}{\tilde\theta}}. \tag{4.11}$$

If

$$\lambda_0 \le \lambda = (R^{n+1}e^{2nR(1-\tilde\theta)})^{\frac{1}{2\beta\tilde\theta}} \le (e^{R(n+1)}e^{2nR(1-\tilde\theta)})^{\frac{1}{2\beta\tilde\theta}},$$

then

$$R \ge \frac{2\beta\tilde\theta}{3n+1-2n\tilde\theta}\log\lambda_0 =: R_0.$$

We take $\delta \le \delta_0 < 1$ with

$$\delta_0^\theta \le e^{-e^{KR_0}}.$$

If $\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_* < \delta$ and $R = \frac{1}{K}\log|\log\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^\theta|$, we have $\lambda \ge \lambda_0$. Then

$$\|F\|_{H^{-1}(\mathbb{R}^n)} \lesssim \left(\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^\theta + \frac{1}{K}\log|\log\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^\theta|^{\frac{-1}{\tilde\theta}}\right)^{\frac{\tilde\theta}{2}}. \tag{4.12}$$

Now if $\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_* \ge \delta > 0$, then we have

$$\|F\|_{H^{-1}(\mathbb{R}^n)} \le \frac{M}{\delta^{\frac{\theta\tilde\theta}{2}}}\delta^{\frac{\theta\tilde\theta}{2}} \le \frac{M}{\delta^{\frac{\theta\tilde\theta}{2}}}\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{\frac{\theta\tilde\theta}{2}} \tag{4.13}$$

for some $M>0$. For any $f \in L^\infty(\mathbb{R}^n)$ and $0<\tilde\sigma<1$, we deduce that

$$|f(x)|^{\frac{n}{1-\tilde\sigma}} \le \|f\|^{\frac{n}{1-\tilde\sigma}-2}_{L^\infty(\mathbb{R}^n)}|f(x)|^2$$

for almost every $x\in\mathbb{R}^n$. Then we have

$$\|\gamma_1-\gamma_2\|_{W^{1,\frac{n}{1-\tilde\sigma}}(\mathbb{R}^n)} \lesssim \|\gamma_1-\gamma_2\|^{\frac{2(1-\tilde\sigma)}{n}}_{H^1(\mathbb{R}^n)}.$$

From Morrey embedding theorem, we obtain that

$$\|\gamma_1-\gamma_2\|_{C^{0,\tilde\sigma}(\mathbb{R}^n)} \lesssim \|\gamma_1-\gamma_2\|_{W^{1,\frac{n}{1-\tilde\sigma}}(\mathbb{R}^n)}. \tag{4.14}$$

From (4.2) and (4.12), (4.13) and (4.14), we have

$$\begin{aligned}
\|\gamma_1-\gamma_2\|_{C^{0,\tilde\sigma}(B)} &\lesssim \|\gamma_1-\gamma_2\|_{W^{1,\frac{n}{1-\tilde\sigma}}(B)} \\
&\lesssim \|\gamma_1-\gamma_2\|_{H^1(B)}^{\frac{2(1-\tilde\sigma)}{n}} \\
&\lesssim \|\log\sqrt{\gamma_1}-\log\sqrt{\gamma_2}\|_{H^1(B)}^{\frac{2(1-\tilde\sigma)}{n}} \\
&\lesssim \Big(\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{\theta}+\frac{1}{K}\log|\log\|\tilde\Lambda_{\gamma_1}-\tilde\Lambda_{\gamma_2}\|_*^{\theta}|^{\frac{-1}{\theta}}\Big)^{\frac{\tilde\theta(1-\tilde\sigma)}{n}}. \qquad (4.15)
\end{aligned}$$

References

[1] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal., **27** (1988), 153–172.

[2] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, J. Differential Equations, **84** (1990), 252–272.

[3] G. Alessandrini, *Open issues of stability for the inverse conductivity problem*, J. Inverse Ill-Posed Problems, **15** (2007), 451–460.

[4] R. Brown and R. H. Torres, *Uniqueness in the inverse conductivity problem for conductivities with* $3/2$ *derivatives in* L^p*,* $p>2n$, J. Fourier Anal. Appl., **9** (2003), 563–574.

[5] A. L. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Comm. Partial Differential Equations, **27** (2002), 653–668.

[6] A. Calderón, *On an inverse boundary value problem*, Seminar in Numerical Analysis and its Applications to Continuum Physics (Río de Janeiro: Soc. Brasileira de Matemática), (1980), 65–73.

[7] P. Caro, Andoni García and J. M. Reyes, *Stability of the Calderón problem for less regular conductivities*, J. Differential Equations, **254** (2013), 469–492.

[8] A. Greenleaf, M. Lassas and G. Uhlmann, *The Calderón problem for conormal potentials, I: Global uniqueness and reconstruction*, Comm. Pure Appl. Math., **56** (2003), 328–352.

[9] B. Haberman and D. Tataru, *Uniqueness in Calderón problem with Lipschitz conductivities*, To appear Duke Math. J..

[10] H. Heck, *Stability estimates for the inverse conductivity problem for less regular conductivities*, Comm. Partial Differential Equations, **34** (2009), 107–118.

[11] D. Holder, Electrical Impedance Tomography, Institute of Physics Publishing, Bristol and Philadelphia, (2005).

[12] D. Holder, D. Isaacson, J. Müller and S. Siltanen, editors, Physiol. Meas., no 1. **25** (2003).

[13] H. Heck and J.-N. Wang, *Stability estimates for the inverse boundary value problem by partial Cauchy data*, Inverse Problems, **22** (2006), 1787–1796.

[14] H. Heck and J.-N. Wang, *Optimal stability estimate of the inverse boundary value problem by partial measurements*, Preprint, (2007), arXiv:0708.3289.

[15] J. Jossinet, *The impedivity of freshly excised human breast tissue*, Physiol. Meas., **19** (1998), 61–75.

[16] K. Knudsen, *The Calderón problem with partial data for less smooth conductivities*, Comm. Partial Differential Equations, **31** (2006), 57–71.

[17] C. E. Kenig, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data*, Ann. Math., **165** (2007), 567–591.

[18] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems, **17** (2001), 1435–1444.

[19] A. Nachman, *Reconstructions from boundary measurements*, Ann. Math., **128** (1988), 531–576.

[20] L. Päivärinta, A. Panchenko and G. Uhlmann, *Complex geometical optics for Lipschitz conductivities*, Revista Matematica Iberoamericana, **19** (2003), 57–72.

[21] M. Salo, *Inverse problems for nonsmooth first order perturbations of the Laplacian*, Ann. Acad. Sci. Fenn. Math. Diss., (2004), no. 139, 67, Dissertation, University of Helsinki, Helsinki.

[22] J. Sylvester and G. Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure Appl. Math., **39** (1986), 92–112.

[23] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. Math., **125** (1987), 153–169.

[24] J. Sylvester and G. Uhlmann, *Inverse boundary value problems at the boundary-continuous dependence*, Comm. Pure Appl. Math., **41** (1988), 197–219.

[25] G. Uhlmann, *Developments in inverse problems since Calderón's foundational paper*, Chapter 19 in "Harmonic Analysis and Partial Differential Equations", University of Chicago Press (1999), 295–345, edited by M. Christ, C. Kenig and C. Sadosky.

[26] G. Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems, 25th Aniversary Volume, **25** (2009), 123011.

[27] S. Vessella, *A continuous dependence result in the analytic continuation problem*, Forum Math., **11** (1999), 695–703.

[28] G. Zhang, *Uniqueness in the Calderón problem with partial data for less smooth conductivities*, Inverse Problems, **28** (2012), 105008.

[29] M. S. Zhdanov and G. V. Keller, *The geoelectrical methods in geophysical exploration*, Methods in Geochemistry and Geophysics, **31** (1994), Elsevier.